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Critical exponents for the supersymmetric σ-model

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Abstract. We compute various critical exponents of the supersymmetric σ-model on S^N within the large-N expansion to $O(1/N)$, and deduce the anomalous dimension to three orders in perturbation theory. We demonstrate that unlike the bosonic model there are no discontinuities in the specific heat.

1. Introduction

It is well known that there is a close relation between continuous field theories and thermodynamic systems. For instance the $O(N)$ nonlinear σ-model near two dimensions is equivalent to the Heisenberg model (Brézin and Zinn-Justin 1976). Both models possess a two-phase structure, where in the former, the $O(N)$ symmetry is broken in one phase, but restored in the upper. Near the critical point, in this and other models, physical quantities such as magnetisation, obey simple power law behaviour. More concretely, they depend on the difference in the temperature and critical temperature raised to some power, which is known as the critical exponent for that quantity. The exponents characterise the system and moreover, the principle of universality implies that they depend only on the dimensions of spacetime and any internal symmetry group, but not on the details of the interaction. Further, various exponents are not independent, since simple physical arguments can be constructed which relate them to other exponents. Such relations are known as scaling laws. (For an introduction see, for instance, Amit (1978).)

As the nonlinear σ-model is of interest because of its relation to the Heisenberg model, techniques have been developed by which its critical exponents can be determined (Ma 1973). Perturbatively the model is renormalisable and asymptotically free only in two dimensions. Beyond two dimensions, one finds that the model is renormalisable only within the (non-perturbative) large N expansion, and it is within this scheme that one deduces exponents for spacetime dimensions d, where $2 < d < 4$. In the initial work of Ma (1973) a large set of exponents were determined for models with long and short range, and Coulomb type interactions to $O(1/N)$. As a consequence the scaling laws were checked and found to be valid except for the specific heat, where it was thought that the law was anomalous in various dimensions d_0, where $d_0 = 2 + 2/n$ for $n > 1$, which was also observed by others in a different approach (Abe and Hikami 1973). Further examination revealed, however, that this scaling law was not anomalous (Abe and Hikami 1974). Instead, the 'anomalous' term which appeared, was a consequence of a discontinuity in the specific heat at these dimensions d_0.

A further reason for determining exponents for σ-models is that one can deduce information about the perturbative structure of the field theory to three loops, by computing at most 1-loop graphs within the large N expansion. Clearly this technique avoids having to compute a large number of graphs in perturbation theory. For instance in (Hikami and Brézin 1978), the structure of the 3-loop anomalous dimension for the bosonic model can be easily written down from symmetry, leaving one constant to be determined. If the relevant exponent from the large N expansion is expanded near two dimensions, using the ϵ expansion ($d=2+\epsilon$), then the constant can be determined.

For this reason it is of interest to extend the work of (Ma 1973) to examine the nonlinear σ-model with supersymmetry, and deduce information about the structure of the various renormalisation group functions to three loops and beyond. Indeed the critical exponents in a supersymmetric extension of the σ-model will of course differ due to additional interactions, and cancellations between various graphs in large N. Moreover, as the bosonic sector of the supersymmetric O(N) model incorporates the model used by Ma, we need only compute those additional graphs which involve fermions. Further, we can examine if the specific heat remains discontinuous in the supersymmetric theory at the dimensions d_0. Finally, we remark that the evaluation of exponents in a supersymmetric model is useful, since it is not inconceivable that supersymmetry may be realised in nature in some bose fermi system.

The paper is organised as follows. Section 2 contains a brief survey of the large N properties of the supersymmetric O(N) model, which are required to compute the exponents. This calculation is detailed in section 3, where the results of our next to leading order computation are summarised and compared with the bosonic case in a table at the end of the section. Finally, in section 4 we discuss the structure of the beta function and anomalous dimension to three loops and beyond in perturbation theory.

2. Brief review of the model

As the large N expansion of the supersymmetric O(N) model has been extensively examined (Alvarez 1978, Aref'eva *et al* 1979), we note only the properties relevant to our investigation. First, the model is described by the Lagrangian

$$L = \tfrac{1}{2}(\partial n)^2 + \tfrac{1}{2}\mathrm{i}\bar{\psi}\partial\!\!\!/\psi - \tfrac{1}{2}\sigma\bar{\psi}\psi - \tfrac{1}{2}\sigma^2 n^2 - \tfrac{1}{2}\bar{u}\psi n - \tfrac{1}{2}\lambda(n^2 - 1/g) \qquad (2.1)$$

where n^i and ψ^i are O(N) vectors, all fields are real, the fermions u and ψ^i are Majorana, σ is auxiliary and λ and u are Lagrange multiplier fields. In computing the effective action with (2.1), the λ-multiplet of fields become dynamical and develop propagators which are determined by inverting their two-point functions, which is equivalent to summing a chain of bubbles in perturbation theory. To leading order they are

$$\begin{aligned} \lambda - 2m\sigma &: -\frac{2\mathrm{i}}{NJ(k,m^2)} \\ u &: -\frac{2\mathrm{i}(k\!\!\!/ - 2m)}{NJ(k,m^2)(k^2-4m^2)} \\ \sigma &: -\frac{2\mathrm{i}}{NJ(k,m^2)(k^2-4m^2)} \end{aligned} \qquad (2.2)$$

where

$$J(k,m^2) = -\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\left(\frac{4m^2-k^2}{4}\right)^{-2+d/2} {}_2F_1\left(2-\frac{d}{2},\frac{1}{2};\frac{3}{2};\frac{k^2}{k^2-4m^2}\right) \tag{2.3}$$

and we work in Minkowski throughout. The quantity m is the mass which is generated in the large N expansion (Alvarez 1978, Aref'eva *et al* 1979).

Next we define the exponents of the system similar to (Ma 1973). Denoting the propagators of n^i and λ by $G(k)$ and $\chi(k)$ respectively, then we have at criticality, for small k

$$G(k) \sim k^{-2+\eta} \qquad \chi(k) \sim k^{\lambda} \tag{2.4}$$

and above the critical temperature

$$G(0) \sim (m_0^2 - m_{0c}^2)^{-\gamma} \qquad \chi(0) \sim (m_0^2 - m_{0c}^2)^{-\alpha} \tag{2.5}$$

where m_{0c}^2 is the critical value of m_0^2 and $(m_0^2 - m_{0c}^2)$ is small. Finally, for the three-point function

$$\Gamma_{2s}(k) = \frac{\langle n^2(k/2)\lambda(k)\rangle}{G^2(k/2)} \tag{2.6}$$

we define

$$\Gamma_{2s} \sim k^{\mu}. \tag{2.7}$$

The scaling laws for the various exponents are (Ma 1973, Amit 1978)

$$\frac{1}{\nu} = \frac{2-\eta}{\gamma} \qquad \alpha = -\frac{\gamma\lambda}{2-\eta} \qquad \mu = (2-\eta)(1-\gamma^{-1}) \tag{2.8}$$

which we will verify for (2.1), as a consequence of our calculations. We note that in determining η, λ and μ, we will use $J(k,m^2)$ with $m = 0$, whilst for γ and α, $m \neq 0$, which is apparent from the definitions.

We close this section by noting how the exponents are deduced within the $1/N$ expansion. First, universality implies that they depend only on N and d. Thus

$$\eta = \eta_0 + \frac{\eta_1}{N} + O\left(\frac{1}{N^2}\right) \tag{2.9}$$

where $\eta_i = \eta_i(d)$. So substituting in (2.3), it is easy to see that

$$G(k) \sim k^{-2+\eta_0}\left(1 + \frac{\eta_1}{N}\ln k + O\left(\frac{1}{N^2}\right)\right). \tag{2.10}$$

Hence calculating $G(k)$ for small k within the $1/N$ expansion, and isolating the $\ln k$ contribution, its coefficient will determine η to $O(1/N)$, (Ma 1973).

3. Determination of exponents

We now present the calculation of the various exponents of (2.4), (2.5) and (2.7). Our discussion concentrates on the contributions which the addition of fermions make, since the purely bosonic graphs (n^i, λ) have been treated in detail in (Ma 1973).

3.1. η

The corrections to the n-propagator are deduced from the self-energy corrections $\Sigma(k)$, which are illustrated in figure 1. Then η is determined from

$$G^{-1}(k) = k^2 + \Sigma(k) - \Sigma(0) \tag{3.1}$$

since for small k

$$G^{-1}(k) \sim k^2(1 - \eta \ln k) . \tag{3.2}$$

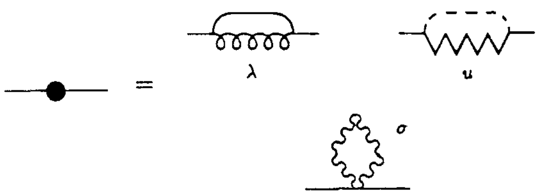

Figure 1. Boson self-energy corrections.

Computing the graphs we find

$$\Sigma(k) = -\frac{2k^2}{N} \int_p \frac{1}{J(p,0)p^2(p+k)^2} \tag{3.3}$$

and note in this case $\Sigma(0) = 0$. The only modification that supersymmetry has been made here compared to the bosonic analysis is an additional factor of p^2 in the denominator of the integral. Clearly, for small k the small p region of (3.3) will yield the $\ln k$ piece (Ma 1973). So with a cut-off μ, and writing the angular integration explicitly in d dimensions

$$\Sigma(k) \sim \frac{\mathrm{i}k^2 2^{d-1}\Gamma(d/2-1)\Gamma(d/2)S_d}{N\pi\Gamma(d-2)} \int_0^\mu \mathrm{d}k \int_0^\pi \mathrm{d}\theta \, \frac{k\sin^{d-2}\theta}{k^2 + 2kp\cos\theta + p^2} \tag{3.4}$$

where

$$S_d = \frac{\Gamma(d-2)}{\Gamma^2(d/2-1)\Gamma(2-d/2)\Gamma(d/2)} . \tag{3.5}$$

The integrations are elementary and yield

$$\eta = 4S_d/N . \tag{3.6}$$

3.2. γ

To determine γ we must consider the self-energy at zero momentum and above the critical point, where there is a non-zero mass. At zero momentum the full mass m is defined via

$$G^{-1}(0) = m^2 \tag{3.7}$$

which may be rewritten in terms of the self-energy $\Sigma(m^2)$ and its critical value via

$$G^{-1}(0) = m_0^2 - m_{0\mathrm{c}}^2 + \Sigma(m^2) - \Sigma(0) \tag{3.8}$$

where $m_{0\mathrm{c}}^2 + \Sigma(0) = 0$ defines $m_{0\mathrm{c}}^2$. Recalling though how γ was defined in (2.5) near criticality, we rewrite (3.8) as (Ma 1973)

$$m^2 - (\Sigma(m^2) - \Sigma(0)) \sim (m^2)^{1/\gamma} \tag{3.9}$$

which allows us to extract $1/\gamma$ from the self-energy corrections. In addition to the graphs of figure 1, this involves the tadpoles of figure 2, which clearly do not contribute to η. At leading order only the first graph of figure 2 gives a contribution, and it is easy to see it is

$$N(m^2)^{-1+d/2}\frac{\Gamma(2-d/2)\Gamma(d/2-1)}{(4\pi)^{d/2}\Gamma(d/2)} \tag{3.10}$$

from which (Ma 1973) $1/\gamma_0 = d/2 - 1$. Hence, to determine γ_1, we need to isolate the $(m^2)^{d/2-1}\ln m^2$ term from the next to leading order self-energy corrections. Careful examination of the graphs of figure 1, reveals they involve logarithmic terms of the form $m^2 \ln m^2$ only, and hence do not contribute. Only the remaining tadpole graph of figure 2 is relevant. Moreover, we note that above criticality the self-energy corrections are now

$$-\frac{2}{N}(p^2 - m^2)\int_k \frac{1}{J(k,m^2)(k^2-4m^2)((k+p)^2-m^2)}$$
$$-\frac{2m^2}{N}\int_k \frac{1}{J(k,m^2)(k^2-4m^2)(k^2-m^2)}\,. \tag{3.11}$$

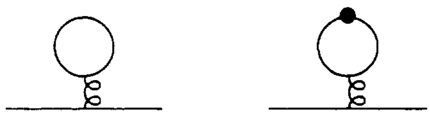

Figure 2. Tadpole corrections to self-energy.

Hence, the correction in figure 2 gives

$$-2\mathrm{i}\int \frac{1}{k^2-4m^2} - 2m^2 J(0,m^2)\mathrm{i}\int \frac{1}{J(k,m^2)(k^2-4m^2)(k^2-m^2)} \tag{3.12}$$

and cutting off the second integral since the small k piece contains the relevant term, we find with (3.10)

$$\gamma^{-1} = \tfrac{1}{2}(d-2)(1+2S_d/N)\,. \tag{3.13}$$

Using the scaling law (2.8) for $1/\nu$, we deduce

$$\nu^{-1} = d - 2 + O(1/N^2) \tag{3.14}$$

from which it is clear that fermion contributions have cancelled the contributions from the boson sector.

3.3. λ

The exponents α and λ relate to the corrections to the bubble sums for the λ field. From (2.3) it is easy to find that $\lambda_0=4-d$, with the corrections given by figure 3. We note that the second graph gives $4\mathrm{i}J(k,0)S_d\ln k$ whilst the self-energy graph gives $\mathrm{i}N\eta J(k,0)S_d\ln k$. The 3-loop boson graph was computed in (Ma 1973), and was found to be $8\mathrm{i}(d-3)J(k,0)S_d\ln k$. This 3-loop graph involves two triangle graphs which have the associated integral

$$T(k,q) = \int_p \frac{1}{p^2(p+k)^2(p+q)^2}\,. \tag{3.15}$$

For latter purposes we note that for large q/k it has the asymptotic form

$$\begin{aligned} T(k,q) &\sim q^{-2}J(k,0)\left(-1+(d-3)\left(\frac{q}{k}\right)^{d-4}\right) \\ &\quad + q^{-2}J(k,0)\sum_{n=1}^{\infty}\left(a_n\left(\frac{q}{k}\right)^{-n} + b_n\left(\frac{q}{k}\right)^{d-4-n}\right). \end{aligned} \tag{3.16}$$

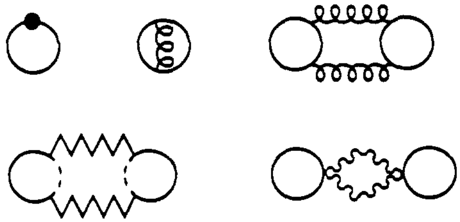

Figure 3. Corrections to the λ two-point function.

This is proved by taking the Mellin transform of T with respect to q, then computing the p integration. Finally, the inverse transform is performed and evaluated from the simple poles in the right half of the transform plane.

The additional contribution to λ comes from the fourth and fifth graphs. However, the final graph does not contain any $\ln k$ piece. The computation of the fourth graph is quite involved and we note various steps. With the (Euclidean space) definition

$$S_\mu = \int_p \frac{p_\mu}{p^2(p+k)^2(p+q)^2} \tag{3.17}$$

the 3-loop integral is

$$\begin{aligned} 4\mathrm{i}\int & \left[(q^2)^2T^2 - 2q\cdot S\,k\cdot S - 2q^2k\cdot ST + 2q\cdot S\,q^2T - q^2k\cdot qT^2 \right. \\ & \left. +(q\cdot S)^2 + q\cdot kS^2\right]\left[J(q,0)J(q-k,0)q^2(q-k)^2\right]^{-1} \end{aligned} \tag{3.18}$$

where we have performed a trace over γ matrices. Clearly, terms in the numerator which behave as $(q/k)^{d-4}$ only, will give the $\ln k$ terms we are seeking. Noting

$$2q\cdot S = J(k-q,0) - J(k,0) - q^2T \tag{3.19}$$

$$2k\cdot S = J(k-q,0) - J(q,0) - k^2T \tag{3.20}$$

it is easy to find that with (3.16)

$$S_\mu \sim q^{-2} J(k,0) q_\mu \left[-\frac{1}{2}(d-4)\left(\frac{q}{k}\right)^{d-4} + \sum_{n=1}^{\infty} b'_n \left(\frac{q}{k}\right)^{d-4-n} \right]$$
$$+ q^{-2} J(k,0) k_\mu \left[\frac{1}{2} + \sum_{n=1}^{\infty} a'_n \left(\frac{q}{k}\right)^{-n} \right] \tag{3.21}$$

for large q/k. Alternatively, one may compute the integral explicitly via the method of Mellin transforms mentioned to yield the same result. As in the case of T, only the leading two terms of S_μ are relevant for our purposes. With (3.16) and (3.21), it is easy to show that the $(q \cdot S)^2$, $q \cdot k S^2$ and $k \cdot q\, q^2 T^2$ terms in (3.18) are irrelevant. Further, rewriting the relevant terms of the numerator with (3.19) and (3.20), we find the $\ln k$ piece is present only in the integrals

$$4\mathrm{i} \int \left[\frac{T(k,q)}{J(q,0)(q-k)^2} - \frac{J(k,0)T(k,q)}{J(q,0)J(q-k,0)(q-k)^2} \right] \tag{3.22}$$

which yields the contribution

$$-8\mathrm{i} J(k,0) S_d (d-2) \ln k \,. \tag{3.23}$$

Thus collecting the various pieces, yields the simple expression

$$\lambda = 4 - d + \mathrm{O}(1/N^2)\,. \tag{3.24}$$

Moreover, the scaling law (2.8) predicts α to be

$$\alpha = \frac{4-d}{2-d} + \mathrm{O}\left(\frac{1}{N^2}\right). \tag{3.25}$$

3.4. α

The graphs required for α are those of figure 3, but considered at zero momentum. These have been computed already within a different context in this model (Gracey 1989), and their sum, without the self-energy subtraction at zero momentum in the first graph of figure 3 may be rewritten as the simple integral

$$-\frac{(d-3)}{2} \int_k \frac{1}{(k^2 - 4m^2)^2} \tag{3.26}$$

which clearly does not yield any $\ln m^2$ term. The remaining self-energy subtraction piece gives

$$\frac{(d-4)}{2} J(0,m^2) \int \frac{1}{J(k,m^2)(k^2-4m^2)(k^2-m^2)} \tag{3.27}$$

which appeared previously in (3.12). After isolating the $\ln m^2$ piece, we note that to leading order

$$\chi(0) = -\frac{2\mathrm{i}}{N J(0,m^2)} \tag{3.28}$$

and with $m^2 \sim (m_0^2 - m_{0c}^2)^\gamma$, we find

$$\alpha = \tfrac{1}{2}\gamma(4-d)(1+2S_d/N) \tag{3.29}$$

which agrees with (3.25).

In the bosonic case the same calculation of α involved the integral (Ma 1973)

$$\int_k \frac{1}{J(k,m^2)(k^2-4m^2)} \tag{3.30}$$

which has in addition to $\ln m^2$ pieces for all d, terms which contribute in dimensions d_0. These arise due to a discontinuity in the specific heat critical amplitude, and do not give additional 'anomalous' contributions to the scaling law (Abe and Hikami 1974). By contrast the integral (3.30) does not occur in the supersymmetric case, and so the specific heat is continuous.

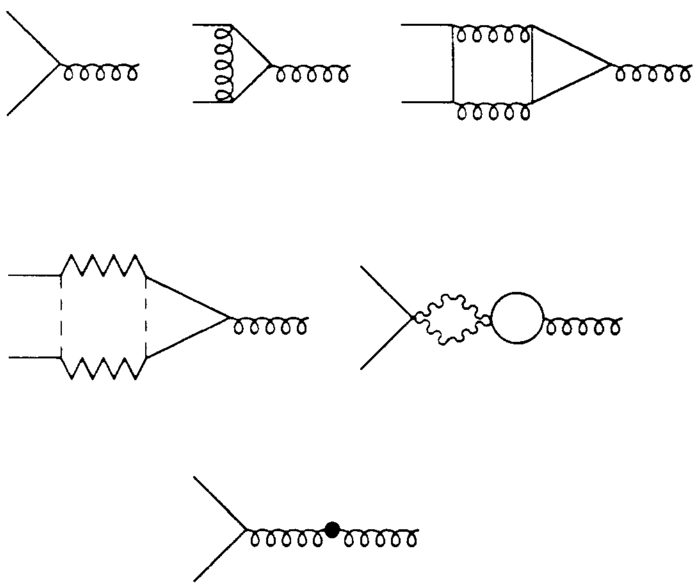

Figure 4. O$(1/N)$ graphs contributing to $\Gamma_{2s}(k)$.

3.5. μ

Finally, the graphs used to determine μ are given in figure 4. As we have detailed the techniques required to extract this exponent in previous calculations, we note only the various cancellations which occur. First, the final graph of figure 4 gives no contribution in the light of (3.24), whilst the fifth graph has no logarithmic piece. Second, only the triangle graph involving fermions is relevant and writing it in terms of T and S_μ, it contains a piece which exactly cancels the third graph. Of the remaining terms the significant integral is

$$-\frac{8}{N^2 J(k,0)} \int_q \frac{1}{J(q,0)q^2(q-k/2)^2} \sim -\frac{16\ln k}{N^2 J(k,0)} S_d. \tag{3.31}$$

Hence, with $8S_d \ln k/(N^2 J(k,0))$ from the second graph, we find

$$\mu = 4 - d - 4S_d/N. \tag{3.32}$$

The results of our calculations are summarised and compared with the bosonic case in table 1.

Table 1. Comparison of the exponents in the bosonic and supersymmetric σ-models in the large-N expansion.

Exponent	Bosonic	Supersymmetric
η	$\frac{4(4-d)}{dN}S_d$	$\frac{4}{N}S_d$
γ	$\frac{2}{d-2}\left(1-\frac{6}{N}S_d\right)$	$\frac{2}{d-2}\left(1-\frac{2}{N}S_d\right)$
ν	$\frac{1}{d-2}\left(1-\frac{8(d-1)}{dN}S_d\right)$	$\frac{1}{d-2}$
λ	$4-d-\frac{16(d-2)(d-1)}{dN}S_d$	$4-d$
α	$\frac{(4-d)}{(2-d)}\left(1+\frac{8(d-1)}{(d-4)N}S_d\right)$	$\frac{(4-d)}{(2-d)}$
μ	$4-d-\frac{4(2d^2-7d+8)}{dN}S_d$	$4-d-\frac{4}{N}S_d$

4. Discussion

We conclude by relating several of the exponents we have to known results, since they are related to various renormalisation group functions. For instance, if g_c is a non-trivial zero of the β-function, $\beta(g_c)=0$, which therefore corresponds to a phase transition, then

$$1/\nu = -\beta'(g_c) \tag{4.1}$$

$$\eta = 2 - d + \gamma(g_c) \tag{4.2}$$

where $\gamma(g)$ is the anomalous dimension. This illustrates the conventional method of determining exponents, since the β function and anomalous dimension can be calculated, in principle, to any order in field theory. Ordinarily, however, such calculations become exceedingly tedious after a few orders. Alternatively, if one knows the exponents from an independent analysis, such as the large-N approach, then one can deduce information on the perturbative structure within this approximation.

We illustrate this point by deducing the 3-loop anomalous dimension for the supersymmetric σ-model. To three loops it takes the form (Hikami and Brézin 1978)

$$\gamma(g) = (N-1)g + \gamma_1(N-1)(N-2)g^3 \tag{4.3}$$

where γ_1 is to be determined. The N dependence is deduced from the fact that on S^1, $\gamma(g)=g$, leading to the factor $(N-2)$. Further, in carrying out a perturbative calculation, in say, the parametrisation of (Brézin and Zinn-Justin 1976), there is always a trace over the O(N) isospin labels in each loop leading to the factor $(N-1)$. The absence of a 2-loop contribution is deduced either from the argument which follows, or by explicit calculation with the parametrisation of S^N of (Brézin and Zinn-Justin 1976). As the boson sector of the supersymmetric model is the bosonic model itself, only the additional contributions to the boson propagator corrections involving fermions need to be computed. To two loops there is only one such extra non-zero graph, which modifies the coupling constant renormalisation, but leaves the 2-loop wavefunction renormalisation as equivalent to that of the bosonic model. Noting that

$$g_c = \frac{d-2}{N-2} + \mathrm{O}\left(\frac{(d-2)^4}{(N-2)^3}\right) \tag{4.4}$$

which we demonstrate later, and substituting (4.4) into (4.3) and expanding in powers of $\epsilon = d-2$, and $1/N$

$$\gamma(g_c) = \epsilon + (\epsilon + \gamma_1\epsilon^3)/N + \mathrm{O}(\epsilon^4) . \tag{4.5}$$

However, performing the same expansion for η, where now

$$S_d = \frac{\epsilon}{4}\exp\left[\sum_{n=3}^{\infty}(2^n - 3 - (-)^n)\zeta(n)\frac{(-\epsilon)^n}{2^n n}\right] \tag{4.6}$$

it is easy to see $\gamma_1 = 0$. Moreover, from (4.6) the 4-loop term of $\gamma(g)$ will be of the form

$$-\tfrac{1}{4}(\zeta(3)N + \gamma_2)(N-1)(N-2)g^4 \tag{4.7}$$

which involves one unknown constant, γ_2.

We have shown that the O($1/N$) correction to ν vanishes in the supersymmetric case, and we now demonstrate this is consistent with the β function to all orders. Its structure can be deduced from simple arguments involving isomorphisms between Grassmann manifolds $M_\alpha(N,p)$, which are symmetric spaces, where

$$M_\alpha(N,p) = G_\alpha(N)/[G_\alpha(N-p) \times G_\alpha(p)] \tag{4.8}$$

and $G_\alpha(N) = \mathrm{O}(N)$, SU($N$) or Sp($N$) when $\alpha=1$, 2 or 4 respectively (Hikami 1982). We note that supersymmetry is unbroken on $M_\alpha(N,p)$ since firstly, even dimensional Grassmann manifolds have positive Euler characteristics (Wolf 1967). Secondly, the argument to construct the Witten index for odd dimensional spheres (Witten 1982), can be readily extended to the remaining orthogonal Grassmann manifolds to show supersymmetry is unbroken on these, too. The approach using isomorphisms was first presented in (Hikami 1981), to deduce the form of the 3-loop β function for bosonic σ-models on symmetric spaces and later extended to four loops (Hikami 1983). We will apply the arguments for the supersymmetric case. In addition to the isomorphisms such as

$$M_1(3,1) \cong M_2(2,1) \qquad M_1(5,1) \cong M_4(2,1) \qquad M_2(4,2) \cong M_1(6,2) \tag{4.9}$$

one allows for the analytic continuation of the classical Lie groups to negative values (Hikami 1981), summarised by the reciprocal relation (Hikami 1982)

$$G_\alpha = G_{4/\alpha}(-\alpha N/2) . \tag{4.10}$$

From these properties and also that the model on S^1 is free, the structure of the β function is determined as follows. If

$$\beta(g) = (d-2)g + \sum_{n=1}^{\infty} \beta_n g^{n+1} \tag{4.11}$$

then the β_n will have terms of the form

$$N^a\left(1-\frac{2}{\alpha}\right)^b (p(N-p))^c \left[\left(1-\frac{1}{\alpha}\right)\left(1-\frac{4}{\alpha}\right)\right]^d \tag{4.12}$$

which satisfies the obvious symmetry $p \to N-p$ from (4.8), and in the light of (4.8) with (4.10) will have to be invariant under the transformations (Hikami 1982)

$$\alpha \to 4/\alpha \qquad N \to -\alpha N/2 \qquad p \to -\alpha p/2 \qquad g \to -2g/\alpha \tag{4.13}$$

implying the constraint

$$a + b + 2c + 2d = n \tag{4.14}$$

at each order in n. Moreover, for the supersymmetric model we use the additional fact that for σ-models on symmetric Kähler manifolds β_n=0 for $n > 1$ (Morozov *et al* 1984), and we note that of (4.8) the two classes $M_2(N,p)$ and $M_1(N,2)$ are Kähler. Thus using *only* these facts, we can immediately write down the 4-loop structure of the 4-loop β function as

$$\begin{aligned}\beta(g) = (d-2)g - \left(N+2-\frac{4}{\alpha}\right)g^2 + b_1\left(1-\frac{2}{\alpha}\right)\left(N+2-\frac{4}{\alpha}\right)\\ \times\left[p(N-p)+2\left(1-\frac{2}{\alpha}\right)\left(N+2-\frac{4}{\alpha}\right)\right]g^5\end{aligned} \tag{4.15}$$

for $M_\alpha(N,p)$, where the constant b_1 is not determined via this method. (We have ignored terms with a factor of $(1-1/\alpha)(1-2/\alpha)(1-4/\alpha)$ which clearly vanishes for the classical Grassmann manifolds.) This result (4.15) can also be deduced from the 4-loop β function of a σ-model on a general Riemannian manifold with one supersymmetry (Grisaru *et al* 1986), which has been calculated via superfield methods, and is a function of the Riemann tensor. Evaluating it for the case of S^N, one finds β_4=$c(N-2)(N-3)$, where c is known and non-zero. We note that the absence of $O(g^3)$ terms in (4.15) for general manifolds was first shown in (Alvarez-Gaumé *et al* 1981). The appearance of a quadratic in N for β_4, rather than a cubic dependence is a particular example of a more general structure for β_n in the supersymmetric model. From the symmetry arguments we mentioned, β_n has at most a N^{n-2} dependence, and not N^{n-1} as occurs in the bosonic case, which can be deduced simply by noting that with $a+c$=$n-1$, $n > 1$

$$\beta_n = aN^{n-1}\left(1-(2/\alpha)\right) + bN^{n-2}p(N-p) + O(N^{n-2})\,. \tag{4.16}$$

(There are no terms $N^n g^{n+1}$ for $n > 1$ from an explicit calculation of the β function in the large N expansion.) As $M_2(N,p)$ and $M_1(N,2)$ are Kähler for all N and p, then a

and b must vanish. Indeed this has to be the case to be in agreement with the exponent ν. From (4.4), an N^{n-1} term in β_n will give a non-zero $O(1/N)$ term in ν, which would disagree with (3.14). We remark that it is the vanishing of β_n $n > 1$ on Kähler spaces which forces N^{n-2} terms in β_n, but of the set of manifolds S^N, only S^2 is Kähler. In the bosonic case N^{n-1} terms are present in β_n, which is clear from table 1, and leads to a more involved structure for β_4 in that model (Hikami 1982).

We close by noting that in the supersymmetric model β_5 involves only two unknown constants, since

$$\begin{aligned}\beta_5 = b_2\bigg[&N^3\left(1-\frac{2}{\alpha}\right)^2 + \frac{N^2}{2}p(N-p)\left(1-\frac{2}{\alpha}\right) - 6\left(1-\frac{2}{\alpha}\right)(p(N-p))^2 \\ &- 13Np(N-p)\left(1-\frac{2}{\alpha}\right)^2 - 22\left(1-\frac{2}{\alpha}\right)^3 p(N-p) + 8\left(1-\frac{2}{\alpha}\right)^5\bigg] \\ &+ b_3\bigg[N^2\left(1-\frac{2}{\alpha}\right)^3 + \frac{5}{2}Np(N-p)\left(1-\frac{2}{\alpha}\right)^2 + 2N\left(1-\frac{2}{\alpha}\right)^4 \\ &+ \left(1-\frac{2}{\alpha}\right)(p(N-p))^2 + 4\left(1-\frac{2}{\alpha}\right)^3 p(N-p)\bigg]. \end{aligned} \tag{4.17}$$

To deduce γ_2, b_2 and b_3 one would either have to perform an explicit perturbative calculation, which would be tedious, or calculate η and ν to $O(1/N^3)$. Techniques for the latter approach have recently been developed and applied to the bosonic model (Vasil'ev *et al* 1981, 1982). We expect that a generalisation to include supersymmetry would be straightforward and probably simplified by a superspace approach. This is a direction we hope to pursue shortly as an insight into the five loop structure of the model will be revealed.

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